A REDUCIBLE CHARACTERISTIC VARIETY IN TYPE A

GEORDIE WILLIAMSON

Abstract. We show that simple highest weight modules for $\mathfrak{sl}_{12}(\mathbb{C})$ may have reducible characteristic variety. This answers a question of Borho-Brylinski and Joseph from 1984. The relevant singularity under Beilinson-Bernstein localization is the (in)famous Kashiwara-Saito singularity. We sketch the rather indirect route via the p-canonical basis, W-graphs and decomposition numbers for perverse sheaves that led us to examine this singularity.

Dedicated to David Vogan on the occasion of his 60^{th} birthday.

1. Introduction

Let $G \supset B \supset T$ denote respectively a complex reductive group, a Borel subgroup and maximal torus. Let W denote its Weyl group, $X = G/B$ the flag variety and T^*X its cotangent bundle. Given $x \in W$ we denote by C_x the corresponding Schubert cell and $T^*_xX \subset T^*X$ its conormal bundle. Let D_X denote the sheaf of algebraic differential operators on X and by $\mathcal{L}_y$ the IC extension of the trivial local system on C_y. We can write the characteristic cycle of $\mathcal{L}_y$ as

$$CC(\mathcal{L}_y) = \sum_{x \in W} m_{x,y}[\overline{T^*_xX}].$$

We have $m_{x,y} \in \mathbb{Z}_{\geq 0}$ and $m_{x,y} = 0$ unless $x \leq y$ in the Bruhat order. The calculation of the multiplicities $m_{x,y}$ is an important and difficult problem. The question we address in this note is:

Question 1.1. (See [BB85, Conjecture 4.5] and [Jos84, §10.2]) Suppose that $G = SL_n(\mathbb{C})$. Is $m_{x,y} = 0$ if $x \neq y$ and x and y lie in the same two-sided Kazhdan-Lusztig cell?

This question is equivalent to asking whether the characteristic variety of a simple highest weight module for $\mathfrak{sl}_n(\mathbb{C})$ is irreducible [BB85, Proposition 6.9]. (A sketch: if $\pi : T^*(G/B) \to \mathfrak{sl}_n(\mathbb{C})^*$ denotes the moment map then the characteristic variety of the global sections of $\mathcal{L}_y$ (a simple highest weight module) agrees with the image of the characteristic variety of $\mathcal{L}_y$ under π [BB85, Corollary 1.5]. The condition on two-sided cells occurs because if $x <_{LR} y$ ($\leq_{LR}$ denotes the Kazhdan-Lusztig two-sided cell preorder) then $\pi(\overline{T^*_xG/B})$ has strictly smaller dimension than $\pi(\overline{T^*_yG/B})$ and hence cannot contribute a reducible component, because characteristic varieties of simple modules are equidimensional [Gab82].) It is known that reducible characteristic varieties occur in other types (e.g. B_2, B_3, C_3) thanks to calculations of Kashiwara and Tanisaki [KT84] and Tanisaki [Tan88].

Kazhdan and Lusztig conjectured (still for $G = SL_n(\mathbb{C})$) that the characteristic varieties of all $\mathcal{L}_y$ are irreducible [KL80a] (that is, that $m_{x,y} = 0$ if $x \neq y$). Of course this would imply an affirmative answer to the above question. However Kashiwara and Saito [KS97] showed that their conjecture was true if $n < 8$ but false for $n \geq 8$. They discovered a singularity (the *Kashiwara-Saito singularity*) which occurs as a normal slice to a Schubert variety in the flag variety of $SL_8(\mathbb{C})$, and for which the characteristic variety is reducible. In their example x and y do not lie in the same two-sided cell, and hence do not provide an example of a reducible characteristic variety of a highest weight module.

In this note we give two permutations $x \leq y$ in S_{12} which lie in the same right cell and such that a normal slice to the Schubert variety corresponding to y along the Schubert cell corresponding to x is isomorphic to the Kashiwara-Saito singularity. This implies that $m_{x,y} \neq 0$, and hence that Question 1.1 has a negative answer.

1.1. **Structure of the paper.** In §2 we discuss the p-canonical basis and prove a result relating characteristic cycle multiplicities and the p-canonical basis. This result is a simple consequence of an observation of Vilonen and the author [VW12]. We then discuss how positivity properties of the p-canonical basis and computer code of Howlett-Nguyen allows one to narrow the search for potential counter-examples. (Indeed, with $12! = 479\ 001\ 600$ Schubert varieties in the flag variety of GL_{12}, the challenge is in the finding rather than the verifying!) In §3 we give the singularity in the GL_{12} flag variety and perform a straightforward calculation to obtain the Kashiwara-Saito singularity.

1.2. **Comments on the literature.** In [Mel93] a proof is proposed for the irreducibility of characteristic varieties in type A. As we have already remarked, this would imply that Question 1.1 has a positive answer. The results of this paper contradict [Mel93, Proposition 3.2] and it is not clear to the author how this proposition follows from the results of [Jos84]. A statement equivalent to [Mel93, Proposition 3.2] is made in the remark on page 54 of [BB85].

1.3. **Acknowledgements.** This paper also owes a significant debt to Leticia Barchini who asked me repeatedly about Question 1.1, and answered questions during and following a visit to the MPI last year. Thanks also to Peter Trapa for some explanations and Anna Melnikov, Yoshihisa Saito and Toshiyuki Tanisaki for useful correspondence. The examples were found using Howlett and Nguyen's software [HN13] for magma [BCP97] which produces the irreducible W-graphs for the symmetric group, implementing an algorithm described in [HN12, §6].

During a visit to MIT last year David Vogan asked me whether the results of [VW12] could produce new examples of reducible characteristic cycles, and asked about Question 1.1. It is a pleasure to dedicate this paper to David, thank him for his many wonderful contributions to Lie theory and to wish him a happy birthday!

2. Motivation from modular representation theory

In this section we sketch the route which led us to consider the singularity in §3.5. We have tried to provide enough details and references that a motivated reader could adapt these techniques to find other interesting (counter)examples. Most of the ideas are already contained in [Wil12], which has more detail than the discussion below.

2.1. **The p-canonical basis.** Let G, B, T be as in the introduction. Let W denote the Weyl group, S its simple reflections, $\leq$ its Bruhat order and ℓ its length function. Consider the flag variety G/B with its stratification by B-orbits (the Schubert stratification):

$$G/B = \bigsqcup_{w \in W} C_w.$$

Fix a field $\Bbbk$ of characteristic $p \geq 0$ and let $D^b_{(B)}(G/B; \Bbbk)$ denote the bounded derived category of constructible sheaves on G/B which are constructible with respect to the Schubert stratification. For $w \in W$ denote by $\mathbf{IC}(w; \Bbbk)$ the intersection cohomology sheaf and $\mathcal{E}(w; \Bbbk)$ the parity sheaf (for the constant pariversity) [JMW09, Wil12] corresponding to $\overline{C_w}$. We will drop the $\Bbbk$ from the notation if it is clear from the context. If $\Bbbk$ is of characteristic 0 then $\mathcal{E}(w; \Bbbk) = \mathbf{IC}(w; \Bbbk)$.

Let $\mathcal{H}$ denote the Hecke algebra of (W, S). It is a free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_w \mid w \in W\}$ and multiplication determined by

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w), \\ (v^{-1} - v)H_w + H_{sw} & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $\{\underline{H}_w\}$ denote the Kazhdan-Lusztig or "canonical" basis of $\mathcal{H}$. We use the normalizations of [Soe97]. For example $\underline{H}_s = H_s + vH_{id}$.

Given a finite dimensional $\mathbb{Z}$-graded vector space $V = \bigoplus V^i$ let

$$\mathrm{ch} V = \sum \dim_{i \in \mathbb{Z}} V^{-i} v^i \in \mathbb{Z}[v^{\pm 1}]$$

denote its Poincaré polynomial. Given $\mathcal{F} \in D^b_{(B)}(G/B; \Bbbk)$ define

$$\mathrm{ch}\, \mathcal{F} = \sum_{x \in W} \mathrm{ch}\, H^*(\mathcal{F}_x) v^{-\ell(x)} H_x \in \mathcal{H}$$

where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at the point $xB/B \in C_x \subset G/B$. It is a classical theorem of Kazhdan and Lusztig [KL80b] (see also [Spr82]) that if $\Bbbk$ is of characteristic zero then

$$\mathrm{ch}\, \mathbf{IC}(w; \Bbbk) = \underline{H}_w. \tag{2.1}$$

For any $w \in W$ we define

$${}^p\underline{H}_w := \mathrm{ch}\, \mathcal{E}(w; \Bbbk).$$

(One can show that ${}^p\underline{H}_w$ only depends on the characteristic p of $\Bbbk$, which explains the notation.) We call the $\{{}^p\underline{H}_w\}$ the *p-canonical basis* for reasons which the following proposition should make clear:

Proposition 2.1.

i) *${}^p\underline{H}_w = H_w + \sum_{x<w} {}^ph_{x,w} H_x$ with ${}^ph_{x,w} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ (hence $\{{}^p\underline{H}_w \mid w \in W\}$ is a basis),*
ii) *${}^p\underline{H}_w = \sum {}^pm_{x,w} \underline{H}_x$ for self-dual ${}^pm_{x,w} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$,*
iii) *if ${}^pm_{x,w}$ are as in (ii) then ${}^pm_{x,w} = 0$ unless $\mathcal{L}(x) \supset \mathcal{L}(w)$ and $\mathcal{R}(x) \supset \mathcal{R}(w)$ where $\mathcal{L}$ and $\mathcal{R}$ denote left and right descent sets,*
iv) *${}^p\underline{H}_x {}^p\underline{H}_y = \sum {}^p\mu^z_{xy} {}^p\underline{H}_z$ for self-dual ${}^p\mu^z_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$,*
v) *for $p \gg 0$, ${}^p\underline{H}_w = {}^0\underline{H}_w = \underline{H}_w$.*

Sketch of proof. By definition the parity sheaf $\mathcal{E}(w)$ is supported on $\overline{C_w}$ and its restriction to C_w is isomorphic to a shifted constant sheaf. (i) now follows easily from the definition of ch.

Each $\mathcal{E}(w;\mathbb{F}_p)$ admits a lift $\mathcal{E}(w;\mathbb{Z}_p)$, a parity sheaf with coefficients in $\mathbb{Z}_p$. Then $\mathcal{E}(w,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a parity sheaf with coefficients in $\mathbb{Q}_p$, and is hence isomorphic to a direct sum of intersection cohomology complexes. (ii) now follows from (2.1) and the fact that $\mathcal{E}(w;\mathbb{F}_p)$, $\mathcal{E}(w,\mathbb{Z}_p)$ and $\mathcal{E}(w,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ all have the same character (see [Wil12, Theorem 3.10]).

For fixed w the parity sheaf $\mathcal{E}(w;\mathbb{F}_p)$ may be obtained via pull-back from the partial flag variety G/P where $P\supset B$ is the parabolic subgroup determined by $\mathcal{R}(w)\subset S$ (see [JMW09, Proposition 4.10]). Hence $\mathcal{R}(x)\supset\mathcal{R}(w)$ as claimed. The statement for left descent sets follows because ${}^pm_{x,w}={}^pm_{x^{-1},w^{-1}}$ by [Wil12, §3 eq. (4)].

Each parity sheaf admits a lift to the B-equivariant derived category $D^b_B(G/B,\Bbbk)$ where there is a convolution formalism categorifying the multiplication in the Hecke algebra. (iv) then follows because the convolution of two parity sheaves is isomorphic to a direct sum of shifts of parity sheaves [JMW09, Theorem 4.8].

Finally (v) follows from (2.1) and [JMW09, Proposition 2.41] which asserts that $\mathcal{E}(w;\mathbb{F}_p)=\mathbf{IC}(w;\mathbb{F}_p)$ for all but finitely many primes p. □

Warning 2.2. The p-canonical basis depends on the root system of G, not just on its Weyl group. (For example the 2-canonical basis differs in types B_3 and C_3.) Hence one should think about the p-canonical basis as a basis of the Hecke algebra attached to a root system or Cartan matrix rather than a Coxeter system. Kashiwara and Saito have observed the same phenomenon for characteristic cycles [KT84, Example 5.4].

2.2. **The p-canonical basis and decomposition numbers.** We briefly recall the notion of a decomposition number for perverse sheaves. An excellent reference is [Jut09].

Let X denote a complex variety, $Z\subset X$ a locally closed smooth subvariety, and $\mathcal{L}$ a local system of free $\mathbb{Z}$-modules on X. One may consider the intersection cohomology extension[1] $\mathbf{IC}(\overline{Z};\mathcal{L})$. It is a perverse sheaf with $\mathbb{Z}$-coefficients on X. One has

$$\mathbf{IC}(\overline{Z};\mathcal{L})\otimes_{\mathbb{Z}}\mathbb{Q}=\mathbf{IC}(\overline{Z};\mathcal{L}\otimes_{\mathbb{Z}}\mathbb{Q})$$

and so $\mathbf{IC}(\overline{Z};\mathcal{L})$ can be thought of as a $\mathbb{Z}$-form of $\mathbf{IC}(\overline{Z};\mathcal{L}\otimes\mathbb{Q})$. In general,

$$\mathbf{IC}(\overline{Z};\mathcal{L})\otimes^L_{\mathbb{Z}}\mathbb{F}_p\in D^b_c(X;\mathbb{F}_p)$$

is perverse but no longer simple. The decomposition matrix encodes the Jordan-Hölder multiplicities of the simple perverse sheaves occurring in $\mathbf{IC}(\overline{Z};\mathcal{L})\otimes^L_{\mathbb{Z}}\mathbb{F}_p$.

In this paper we will be concerned with the flag variety together with its Schubert stratification, as in §2.1. In this case all the strata are simply connected and the decomposition matrix takes the form $(d_{y,x})_{y,x\in W}$ where

$$d_{y,x}:=[\mathbf{IC}(\overline{C_y};\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_p:\mathbf{IC}(\overline{C_x};\mathbb{F}_p)].$$

The relation between the characters of the parity sheaves (i.e. the p-canonical basis) and the decomposition matrix is subtle. For example, recent papers of Achar and Riche [AR14a, AR14b] prove that knowledge of the p-canonical basis gives (a q-refinement of) the decomposition matrix for perverse sheaves on the Langlands dual flag variety.

[1]For the perversity p, see [Jut09].

Here we will be concerned with a much more limited but simpler relationship. Roughly it says that the first time the p-canonical basis differs from the canonical basis corresponds to the first non-trivial decomposition number (see Proposition 2.1 and above for notation):

Proposition 2.3. *Fix $y \in W$ and suppose that $x < y$ is maximal in the Bruhat order such that ${}^p m_{x,y} \neq 0$. If ${}^p m_{x,y} \in \mathbb{Z}$ then $d_{y,x} = {}^p m_{x,y}$.*

Proof. Fix x and y are in the proposition. Set

$$X = \bigsqcup_{z \geq x} BzB/B, \qquad Z = BxB \subset X, \qquad U = X \setminus U$$

and denote by i (resp. j) the closed (resp. open) embedding of Z (resp. U) into X. Note that X is open in G/B.

For $\Bbbk \in \{\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p\}$ let $\mathbf{IC}_\Bbbk$ (resp. $\mathcal{E}_\Bbbk$) denote the intersection cohomology (resp. parity) sheaf corresponding to the stratum $ByB/B \subset X$. We have $\mathcal{E}_{\mathbb{Q}_p} \cong \mathbf{IC}_{\mathbb{Q}_p}$ and our assumptions guarantee that $\mathcal{E}$ is perverse with

$$\mathbf{IC}_{\Bbbk|U} \cong \mathcal{E}_{\Bbbk|U}.$$

Hence we need to examine the difference between $\mathbf{IC}_{\mathbb{F}_p}$ and $\mathcal{E}_{\mathbb{F}_p}$ over the closed stratum Z.

Our main tool will be [JMW09, Lemma 2.18] which gives a bijection between isomorphism classes of extensions of a fixed $\mathcal{F}$ on U to X, and isomorphism classes of distinguished triangles on Z of the form

$$A \to i^* j_* \mathcal{F} \to B \xrightarrow{[1]} . \tag{2.2}$$

If $\mathcal{F}'$ is such an extension then A and B are given by

$$i^* \mathcal{F}' \cong A \quad \text{and} \quad i^! \mathcal{F}' \cong B[-1]. \tag{2.3}$$

Let us examine the triangle corresponding to the extension $\mathcal{E}_{\mathbb{Z}_p}$ of $\mathbf{IC}_{\mathbb{Z}_p|U}$. It has the form

$$A \to i^* j_* (\mathbf{IC}_{\mathbb{Z}_p|U}) \to B \xrightarrow{[1]} \tag{2.4}$$

Because Z is contractible we can view (2.4) as a distinguished triangle of $\mathbb{Z}_p$-modules. By (2.3) and the fact that $\mathcal{E}$ is a parity sheaf we deduce:

(1) $H^m(A)$ and $H^m(B)$ are free $\mathbb{Z}_p$-modules;
(2) $H^m(A) = 0$ if $m - \ell(y)$ is odd, and $H^m(B) = 0$ if $m - \ell(y)$ is even.

The assumptions of the proposition and (2.3) guarantee that

(3) $H^m(A)$ vanishes for $m > -\ell(x)$ and $H^m(B)$ vanishes for $m < -\ell(x) - 1$;
(4) $H^{-\ell(x)}(A)$ is free of rank ${}^p m_{x,y}$ (in particular $\ell(y) - \ell(x)$ is even).

Because $\mathbb{Z}_p$ is hereditary, each of the terms in (2.4) is isomorphic to its cohomology. Hence we can turn the triangle and rewrite it as

$$H^*(B)[-1] \to H^*(A) \to H^*(i^* j_* \mathcal{F}) \xrightarrow{[1]} .$$

By (3) above the only non-zero map component of the first map is

$$\alpha : H^{-\ell(x)-1}(B) \to H^{-\ell(x)}(A).$$

Because $\mathcal{E}$ is indecomposable, α does not map any summand of $H^{-\ell(x)-1}(B)$ isomorphically onto a summand of $H^{-\ell(x)}(A)$ by [JMW09, Lemma 2.21]. In other words, $\alpha \otimes_{\mathbb{Z}_p} \mathbb{F}_p = 0$. On the other hand, we have

$$\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbf{IC}_{\mathbb{Q}_p} \oplus \mathbf{IC}(Z)^{\oplus({}^p m_{x,y})}$$

and hence α is an isomorphism over $\mathbb{Q}_p$. In other words, $\ker \alpha = 0$ and the domain and codomain of α are free of the same rank.

By the long exact sequence of cohomology we deduce that:

$$H^m(i^* j_* \mathbf{IC}_{\mathbb{F}_p|U}) = \begin{cases} H^m(A) \otimes \mathbb{F}_p & \text{if } m < \ell(x) - 1, \\ H^{-\ell(x)}(A) \otimes \mathbb{F}_p & \text{if } m = -\ell(x) - 1 \text{ or } m = -\ell(x), \\ H^m(B) \otimes \mathbb{F}_p & \text{if } m > -\ell(x). \end{cases}$$

$$H^m(i^* j_* \mathbf{IC}_{\mathbb{Q}_p|U}) = \begin{cases} H^m(A) \otimes \mathbb{F}_p & \text{if } m < \ell(x) - 1, \\ 0 & \text{if } m = -\ell(x) - 1 \text{ or } m = -\ell(x), \\ H^m(B) \otimes \mathbb{F}_p & \text{if } m > -\ell(x). \end{cases}$$

By the Deligne construction [BBD82, Proposition 2.1.11] we have

$$i^* \mathbf{IC}_{\Bbbk} = i^* \tau_{<-\ell(x)} j_* \mathbf{IC}_{\Bbbk} = \tau_{<-\ell(x)} i^* j_* \mathbf{IC}_{\Bbbk}.$$

where $\tau_{<m}$ denotes truncation. Hence if χ denotes the Euler characteristic at any point in Z we have

$$\chi(\mathbf{IC}_{\mathbb{F}_p}) = \chi(\mathbf{IC}_{\mathbb{Q}_p}) - (-1)^{-\ell(x)}({}^p m_{x,y}).$$

Now we are done: if we write

$$[\mathbf{IC}_{\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p] = [\mathbf{IC}_{\mathbb{F}_p}] + a[\mathbf{IC}(Z)]$$

in the Grothendieck group of $\mathbb{F}_p$-perverse sheaves on X then taking Euler characteristics over Z yields $a = {}^p m_{x,y}$ as claimed. $\square$

As in the introduction we write $m_{x,y}$ for the the characteristic cycle multiplicities. The following is an immediate consequence of the previous proposition, and [VW12, Theorem 2.1].

Corollary 2.4. *Suppose that $x < y$ are as in the previous proposition. Then*

$$m_{x,y} \geq {}^p m_{x,y}.$$

2.3. **Searching for a counter-example.** Consider the following variant of Question 1.1 (with notation as in Proposition 2.1):

Question 2.5. Suppose that $G = SL_n(\mathbb{C})$ and let p be a prime. Is ${}^p m_{x,y} = 0$ if $x \neq y$ and x and y lie in the same two-sided cell?

It will become clear below that a positive answer to Question 1.1 implies a positive answer to Question 2.5. Question 2.5 is also important for modular representation theory, with connections to Lusztig's conjecture around the the Steinberg weight [Soe00], amongst other things.

One can show (using Soergel calculus [EW13] or Schubert calculus [HW14]) that the counter-example in §3 also gives a counter-example to Question 2.5. We found the examples by pursuing a naive idea, which is the main theme of [Wil12]: the p-canonical basis has remarkable positivity properties (summarized in Proposition 2.1) and these positivity properties are enough to rule out many potential counter-examples.

Asume that W is an arbitrary Weyl group. For any left cell $C \subset W$ we can consider the corresponding cell module

$$M_C = \bigoplus_{x \in C} \mathbb{Z}[v^{\pm 1}] M_x := \bigoplus_{x \leq_L C} \mathbb{Z}[v^{\pm 1}] \underline{H}_x / (\bigoplus_{x <_L C} \mathbb{Z}[v^{\pm 1}] \underline{H}_x).$$

The $\mathcal{H}$-module structure in the basis $\{M_x\}$ is encoded in the W-graph of C. Fix a prime p and assume that the p-canonical basis satisfies:

$$\text{for all } y \in C \text{ if } {}^p m_{x,y} \neq 0 \text{ then } x \leq_L y. \tag{2.5}$$

Then we may define ${}^p M_y$ as the image of ${}^p \underline{H}_y$ in M_C and obtain in this way a p-canonical basis for the cell module M_C. By Proposition 2.1 it satisfies the following properties:

(1) (positive upper-triangularity) we have
$$ {}^p M_y = M_y + \sum_{C \ni x < y} {}^p m_{x,y} M_x \text{ with } {}^p m_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \text{ self-dual;}$$
(2) (positive structure constants) for any $x \in W$,
$$ {}^p \underline{H}_x \cdot {}^p M_y \in \bigoplus_{z \in C} \mathbb{Z}_{\geq 0}[v^{\pm 1}]({}^p M_z).$$

Example 2.6. Suppose that W is of type B_2 with simple reflections s, t. Consider the left cell $C = \{s, ts, sts\}$. The W-graph is:

$$\{s\} \text{ ——— } \{t\} \text{ ——— } \{s\}$$

In this case there are two possible bases for M_C satisfying (1) and (2). The first is the Kazhdan-Lusztig basis $\{M_x\}$. The second is the basis $\{M'_x\}$ with $M'_x = M_x$ for $x \in \{s, ts\}$ and $M'_{sts} := M_{sts} + M_s$. In this case M' agrees with the image of the 2-canonical basis for B_2 (for an appropriate choice of long and short root).

Now assume that W is of type A_{n-1}. In this case two sided cells are parametrized by partitions λ of n. Also, all left cells in fixed two sided are irreducible and afford isomorphic (based) representations of the Hecke algebra $\mathcal{H}$.

Lemma 2.7. *Let λ be a partition of n and $E_\lambda \subset W$ the corresponding two-sided cell. Then there exists a left cell $C \subset E_\lambda$ satisfying* (2.5).

Sketch of proof. Let w_λ denote the longest element of the standard parabolic subgroup $W_\lambda \subset W$ determined by λ. Then $w_\lambda \in E_\lambda$. We claim that the left cell C containing w_λ satisfies (2.5). Firstly, ${}^p \underline{H}_{w_\lambda} = \underline{H}_{w_\lambda}$ by (i) and (iii) of Proposition 2.1 and a simple induction then shows that

$$ {}^p \underline{H}_y = \sum_{x \leq_L w_\lambda} {}^p m_{x,y} \underline{H}_x.$$

for all $y \in C$. Hence (2.5) holds. □

It follows that any left cell representation in type A admits a p-canonical basis satisfying the above positive conditions. One can apply computer searches in order to isolate potential counter-examples and then use Soergel calculus [EW13] or Schubert calculus [HW14] to check whether one has indeed found a counter-example.

In order to implement this approach one needs the W-graphs of the left cell representations in type A. These are provided by the wonderful code of Howlett and Nguyen [HN13] for magma [BCP97].

Remark 2.8.

(1) Using the recent results of Achar-Riche [AR14a, AR14b] one can show that if there is a counter-example for a left cell corresponding to λ then there is also a counter-example for the left cell corresponding to the transposed partition λ^t. This allows one to roughly halve the number of left cells which one needs to consider. Experimentally, the above positivity properties are more restrictive in left cells corresponding to partitions "near the top" of the dominance order. (For example for S_4 there is only one solution for the left cell corresponding to the partition $(3,1)$, whereas there are two for the partition $(2,1,1)$.)

(2) Lusztig has given a beautiful description of the J-ring for a fixed two-sided cell in S_n as a (based) matrix ring. Using this result one can show that if the p-canonical basis is trivial (i.e. equal to the image of the Kazhdan-Lusztig basis) in a fixed left cell then Question 2.5 has a positive answer for that two-sided cell.

(3) The above methods yielded another counter-example to Question 2.5, this time in GL_{13}:

$$x = 12132156543765438798765ba98c,$$
$$y = 121321546543765438798765aba9876cba98$$

Here we write x and y as words in the simple transpositions $1,\dots 9, a, b, c$ of S_{13}. Yoshihisa Saito has informed me that in this case one also obtains the Kashiwara-Saito singularity as a normal slice.

3. Two realisations of the Kashiwara-Saito singularity

3.1. **Notation.** Fix a positive integer $n \geq 1$.

Let S_n denote the symmetric group, which we regard as permutations of the set $\{1,\dots,n\}$. We view S_n as a Coxeter group with Coxeter generators the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. We write ℓ for the length function on S_n and $\leq$ for the Bruhat order.

We will usually write permutations in "string notation" i.e. we write $x = x_1x_2\dots x_n$ to mean that x is the permutation in S_n which sends $1 \mapsto x_1$, $2 \mapsto x_2$ etc. To avoid confusion when using string notation we extend our alphabet of digits $1,\dots,9$ by the letters $a, b\dots$ with $a = 10$, $b = 11\dots$.

Let $G = GL_n(\mathbb{C})$ denote the general linear group of invertible complex matrices. Given $x = x_1x_2\dots x_n \in S_n$ we will denote by $\dot{x}$ the corresponding permutation matrix. That is $\dot{x}(e_i) = e_{x_i}$ if $e_1, e_2, \dots, e_n$ denotes the standard basis of $\mathbb{C}^n$.

Let $B \subset G$ denote the Borel subgroup of upper triangular matrices. Let G/B denote the flag variety. Given $y \in S_n$ we denote by

$$C_y = B\dot{y}B/B \subset G/B$$

its Schubert cell and by Z_y the corresponding Schubert variety:

$$Z_y := \overline{B\dot{y}B/B} = \overline{C_y}.$$

3.2. **Equations for slices to Schubert cells.** We recall how to explicitly write down equations for slices to Schubert cells in Schubert varieties. Everything here can be checked reasonably easily by hand with the (possible) exception of the fact that the equations (3.1) are complete.

Let $U_-, U \subset GL_n(\mathbb{C})$ the subgroups of unipotent lower and upper triangular matrices respectively. The natural map $U_- \to G/B$ is an open immersion, giving a coordinate patch isomorphic to $\mathbb{A}^{\binom{n}{2}}$ around the base point $B \in G/B$. Hence for any $x \in S_n$ the natural map $\pi : \dot{x}U_- \to G/B$ gives a coordinate patch around $\dot{x}B \in G/B$. For a permutation $x = x_1 \dots x_n$ we have:

$$\dot{x}U_- = \{g = (g_{i,j}) \in GL_n(\mathbb{C}) \mid g_{x_i,i} = 1 \text{ and } g_{x_i,j} = 0 \text{ for } j > i\}.$$

For $y \in S_n$ the inverse image $\pi^{-1}(Z_y) \subset \dot{x}U_-$ is given by the equations (see [Ful92], [WY08, §3.2] and [WY12, §2.2]):

$$\operatorname{rank}((g_{i,j})_{a \le i \le n, 1 \le j \le b}) \le \operatorname{rank}((\dot{y}_{i,j})_{a \le i \le n, 1 \le j \le b}) \quad \text{for all} \quad 1 \le a, b \le n. \tag{3.1}$$

We have

$$\pi^{-1}(C_x) := \{g \in \dot{x}U_- \mid g_{i,j} = 0 \text{ for } i > x_j\}.$$

Hence if we set

$$N_x = \{g \in \dot{x}U_- \mid g_{i,j} = 0 \text{ for } i < x_j\}$$

then N_x is a normal slice to C_x in $\dot{x}U_-$. Hence the singularity of Z_y along C_x is given by $N_x \cap \pi^{-1}(Z_y)$ which is given by intersecting the linear equations describing N_x with the equations (3.1).

Exercise 3.1. Perhaps an example will help decipher the notation. Consider $n = 4$ and let $x = 2143$ and $y = 4231$. We have

$$N_x = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ c & d & 1 & 0 \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{C} \right\}$$

and the rank conditions (3.1) reduce in this case to the single equation $ad - bc = 0$.

3.3. **The Kashiwara-Saito singularity.** Let $M_2(\mathbb{C})$ denote the space of 2×2-complex matrices with coefficients in $\mathbb{C}$. Consider the space S of matrices $M_i \in M_2(\mathbb{C})$ for $i \in \mathbb{Z}/4\mathbb{Z}$ satisfying the two conditions:

$$\det M_i = 0 \text{ for } i \in \mathbb{Z}/4\mathbb{Z}, \tag{3.2}$$

$$M_i M_{i+1} = 0 \text{ for } i \in \mathbb{Z}/4\mathbb{Z}. \tag{3.3}$$

Clearly S is an affine variety. One can show that it is irreducible of dimension 8. We call S (or more precisely the singularity of S at $0 := (0, 0, 0, 0) \in S$) the *Kashiwara-Saito* singularity. In [KS97] it is shown that the conormal bundle to 0 is a component of the characteristic cycle of the intersection cohomology D-module on S. In particular, the characteristic cycle is reducible.

3.4. **Realisation in GL_8.** Now let $n = 8$ and consider the permutations

$$u := 21654387, \quad v := 62845173.$$

Then u is the maximal element in the standard parabolic subgroup $\langle s_1, s_3, s_4, s_5, s_7 \rangle$ of length $\ell(u) = 8$. We have $u \le v$ and $\ell(v) = 16$.

The following is stated without proof in [KS97, §8.3]. We give the proof here because it is a good warm-up for the calculation in GL_{12} which we need to perform next.

Proposition 3.2. *The singularity of Z_v along C_u is isomorphic to S.*

Proof. If $J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have (as a matrix of block 2×2-matrices):

$$N_u := \left\{ \begin{pmatrix} J & 0 & 0 & 0 \\ A_1 & 0 & J & 0 \\ A_2 & J & 0 & 0 \\ A_0 & A_3 & A_4 & J \end{pmatrix} \middle| A_i \in M_2(\mathbb{C}) \right\}.$$

Now

$$\dot{v} = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

and after some checking one sees that the rank conditions (3.1) reduce to the following equations:

$$A_0 = 0, \tag{3.4}$$

$$\operatorname{rank} \begin{pmatrix} A_2 & J \\ 0 & A_3 \end{pmatrix} \leq 2, \tag{3.5}$$

$$\operatorname{rank} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \leq 1, \tag{3.6}$$

$$\operatorname{rank} \begin{pmatrix} A_3 & A_4 \end{pmatrix} \leq 1, \tag{3.7}$$

$$\operatorname{rank} \begin{pmatrix} A_1 & 0 & J \\ A_2 & J & 0 \\ 0 & A_3 & A_4 \end{pmatrix} \leq 4. \tag{3.8}$$

Now

$$\begin{pmatrix} A_2 & J \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ -JA_2 & J \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_3JA_2 & A_3J \end{pmatrix}$$

and so (3.5) is equivalent to $A_3JA_2 = 0$. Similarly one may show that together (3.5) and (3.8) are equivalent to the conditions

$$A_3JA_2 = 0 \text{ and } A_4JA_1 = 0. \tag{3.9}$$

If we let $K := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then (3.6) is equivalent to the conditions:

$$A_2KA_1^t = 0 \text{ and } \det A_1 = \det A_2 = 0.$$

Similarly, (3.7) is equivalent to the conditions

$$A_4^tKA_3 = 0 \text{ and } \det A_3 = \det A_4 = 0$$

Hence if we set

$$A_1' := A_1^tJ, \quad A_2' := A_2K, \quad A_3' := A_3J, \quad A_4' := A_4^tK.$$

Then the relations become

$$\det A_i' = 0 \text{ for } i \in \{1,2,3,4\},$$
$$A_2'A_1' = A_3'A_2' = A_4'A_3' = A_1'A_4' = 0.$$

This is clearly isomorphic to the Kashiwara-Saito singularity. $\square$

3.5. **Realisation in GL_{12}.** We will see how to realise the Kashiwara-Saito singularity as a normal slice in Z_y to a Schubert cell C_x. This time x and y belong to the same right cell.

Now let $n = 12$. Consider the permutations in S_{12}:

$$x = 438721a965cb, \quad y = 4387a2c691b5.$$

(Remember that we use string notation and $a = 10$, $b = 11, c = 12$.) The Robinson-Schensted P and Q symbols of x are

$$P(x) = \begin{matrix} 1 & 5 & 9 & b \\ 2 & 6 & a & c \\ 3 & 7 & & \\ 4 & 8 & & \end{matrix} \quad \text{and} \quad Q(x) = \begin{matrix} 1 & 3 & 7 & b \\ 2 & 4 & 8 & c \\ 5 & 9 & & \\ 6 & a & & \end{matrix}.$$

The P and Q symbols of y are

$$P(y) = \begin{matrix} 1 & 5 & 9 & b \\ 2 & 6 & a & c \\ 3 & 7 & & \\ 4 & 8 & & \end{matrix} \quad \text{and} \quad Q(y) = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 9 & b \\ 6 & 8 & & \\ a & c & & \end{matrix}.$$

In particular, we conclude that x and y are in the same two-sided cell (even the same right cell).

Reduced expressions for x and y are given by:

$$x = s_b s_5 s_6 s_7 s_8 s_9 s_5 s_6 s_7 s_8 s_7 s_1 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_1$$
$$y = s_5 s_6 s_7 s_8 s_9 s_a s_b s_a s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_7 s_8 s_4 s_5 s_6 s_7 s_1 s_2 s_3 s_4 s_5 s_3 s_1$$

We have $x \leq y$, $\ell(x) = 22$ and $\ell(y) = 30$.

Recall the Kashiwara-Saito singularity S from the previous section.

Proposition 3.3. *The singularity of Z_y along C_x is isomorphic to S.*

Proof. The normal slice N_x to C_x inside the full flag variety is given by the space of matrices

$$\begin{pmatrix} 0 & 0 & J & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & A_1 & 0 & J & 0 \\ B_2 & J & 0 & 0 & 0 & 0 \\ B_3 & B_5 & A_2 & J & 0 & 0 \\ B_4 & B_6 & B_7 & A_3 & A_4 & J \end{pmatrix}$$

where $J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as above and the A_i, B_i are in $M_2(\mathbb{C})$. Now, after some checking one sees that the rank conditions (3.1) give that the intersection of N_x

and Z_y is cut out by the equations:

$$B_i = 0 \tag{3.10}$$

$$\operatorname{rank}\begin{pmatrix} A_3 & A_4 \end{pmatrix} \le 1, \tag{3.11}$$

$$\operatorname{rank}\begin{pmatrix} 0 & A_1 \\ J & 0 \\ 0 & A_2 \end{pmatrix} \le 3 \quad \Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \le 1, \tag{3.12}$$

$$\operatorname{rank}\begin{pmatrix} A_2 & J \\ 0 & A_3 \end{pmatrix} \le 2, \tag{3.13}$$

$$\operatorname{rank}\begin{pmatrix} 0 & A_1 & 0 & J \\ J & 0 & 0 & 0 \\ 0 & A_2 & J & 0 \\ 0 & 0 & A_3 & A_4 \end{pmatrix} \le 6 \quad \Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} A_1 & 0 & J \\ A_2 & J & 0 \\ 0 & A_3 & A_4 \end{pmatrix} \le 4. \tag{3.14}$$

Looking at the proof of Proposition 3.2 it is now clear that $N \cap Z_y \cong S$, the Kashiwara-Saito singularity. $\square$

References

[AR14a] P. Achar and S. Riche. Modular perverse sheaves on flag varieties I: tilting and parity sheaves. preprint arXiv:1401.7245, 2014.

[AR14b] P. Achar and S. Riche. Modular perverse sheaves on flag varieties II: Koszul duality and formality. preprint arXiv:1401.7256, 2014.

[BB85] W. Borho and J.-L. Brylinski. Differential operators on homogeneous spaces. III: Characteristic varieties of Harish Chandra modules and of primitive ideals. *Invent. Math.*, 80:1–68, 1985.

[BBD82] A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[EW13] B. Elias and G. Williamson. Soergel calculus. preprint arXiv:1309.0865, 2013.

[Ful92] W. Fulton. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. *Duke Math. J.*, 65(3):381–420, 1992.

[Gab82] O. Gabber. Equidimensionalité de la variété caractéristique. Exposé de O. Gabber redigé par T. Levasseur, 1982.

[HN12] R. B. Howlett and V. M. Nguyen. W-graph ideals. *J. Algebra*, 361:188–212, 2012.

[HN13] R. Howlett and V. M. Nguyen. W-graph magma programs. software available at http://www.maths.usyd.edu.au/u/bobh/magma/, 2013.

[HW14] X. He and G. Williamson. Soergel calculus and Schubert calculus. in preparation, 2014.

[JMW09] D. Juteau, C. Mautner, and G. Williamson. Parity sheaves. preprint arXiv:0906.2994, to appear in JAMS, 2009.

[Jos84] A. Joseph. On the variety of a highest weight module. *J. Algebra*, 88:238–278, 1984.

[Jut09] D. Juteau. Decomposition numbers for perverse sheaves. *Ann. Inst. Fourier (Grenoble)*, 59(3):1177–1229, 2009.

[KL80a] D. Kazhdan and G. Lusztig. A topological approach to Springer's representations. *Adv. Math.*, 38:222–228, 1980.

[KL80b] D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.

[KS97] M. Kashiwara and Y. Saito. Geometric construction of crystal bases. *Duke Math. J.*, 89(1):9–36, 1997.

[KT84] M. Kashiwara and T. Tanisaki. The characteristic cycles of holonomic systems on a flag manifold related to the Weyl group algebra. *Invent. Math.*, 77:185–198, 1984.

[Mel93] A. Melnikov. Irreducibility of the associated varieties of simple highest weight modules in $\mathfrak{sl}(n)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(1):53–57, 1993.

[Soe97] W. Soergel. Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997.

[Soe00] W. Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *J. Pure Appl. Algebra*, 152(1-3):311–335, 2000.

[Spr82] T. A. Springer. Quelques applications de la cohomologie d'intersection. In *Bourbaki Seminar, Vol. 1981/1982*, volume 92 of *Astérisque*, pages 249–273. Soc. Math. France, Paris, 1982.

[Tan88] T. Tanisaki. Characteristic varieties of highest weight modules and primitive quotients. Representations of Lie groups: analysis on homogeneous spaces and representations of Lie groups, Proc. Symp., Kyoto/Jap. and Hiroshima/Jap. 1986, Adv. Stud. Pure Math. 14, 1-30 (1988)., 1988.

[VW12] K. Vilonen and G. Williamson. Characteristic cycles and decomposition numbers. preprint arXiv:1208.1198, to appear in Math. Res. Let., 2012.

[Wil12] G. Williamson. Modular intersection cohomology complexes on flag varieties. *Math. Z.*, 272(3-4):697–727, 2012. (with an appendix by Tom Braden).

[WY08] A. Woo and A. Yong. Governing singularities of Schubert varieties. *J. Algebra*, 320(2):495–520, 2008.

[WY12] A. Woo and A. Yong. A Gröbner basis for Kazhdan-Lusztig ideals. *Amer. J. Math.*, 134(4):1089–1137, 2012.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `geordie@mpim-bonn.mpg.de`
URL: `http://people.mpim-bonn.mpg.de/geordie/`